Lecture 5

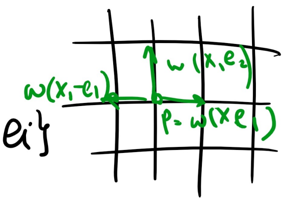
Random walks in Random environment for $d \geq 2$

Walking on $\mathbb{Z}^d, d \geq 2$, nearest neighbour walk

P - environment measure. Generally IID,

uniformly elliptic: $\exists \varepsilon > 0, P(w(0,e) > \varepsilon) = 1 \ \forall e \in \{\pm e_i\}$

elliptic: $P(w(0,e) > 0) = 1$



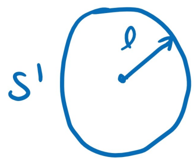
P_ω^x - quenched RW - starting at x and walking in ω.

P^x - annealed measure - also average over ω

Question 1: Recurrence/Transience

We study directional transience.

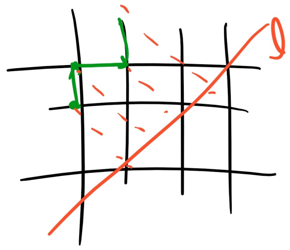
I.e., fix $\ell \in S^{d-1}$ Study $X_n \cdot \ell$



claim: $\mathbb{P}^0(\limsup X_n \cdot \ell \text{ is finite}) = 0$

saw this last time

for any unif. elliptic ω



So we have 3 options:

① $\lim_{n\to\infty} X_n \cdot \ell = +\infty$ call this event A_ℓ - directional transience in direction ℓ.

② $\lim_{n\to\infty} X_n \cdot \ell = -\infty$ call this event $A_{-\ell}$

③ $\limsup_{n\to\infty} X_n \cdot \ell = +\infty$, $\liminf_{n\to\infty} X_n \cdot \ell = -\infty$ Call this O_ℓ - projection to ℓ is neighbourhood recurrent.

By Claim: $\mathbb{P}^x_\omega(A_\ell \cup A_{-\ell} \cup O_\ell) = 1$

unif. elliptic $\Longrightarrow P^x(A_\ell \cup A_{-\ell} \cup O_\ell) = 1$

Important open question: Is $\mathbb{P}^0(A_\ell) \in \{0,1\}$?

What we know is only:

① Thm: (Kalikow 1981 IID, uniformly elliptic. Extended to IID elliptic by Merkl-Zerner 2001)

$P^0(O_\ell) \in \{0,1\}$. This is equivalent to $\mathbb{P}^0(A_\ell \cup A_{-\ell}) \in \{0,1\}$

② In $d=2$,

Thm (Merkl-Zerner 2001, IID elliptic): $\mathbb{P}^0(A_\ell) \in \{0,1\}$

They also showed that there exists a stationary and ergodic P which is elliptic. for which $\mathbb{P}^0(A_\ell) = \mathbb{P}(A_{-\ell}) = \frac{1}{2}$

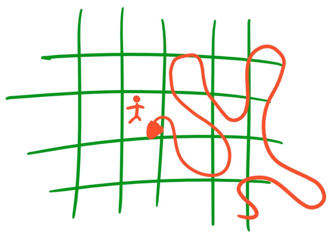
We proceed to prove Kalikow's 0-1 law
In the IID Unif. elliptic case:

Take $\ell = (1,0,0,\ldots)$ for notational Simplicity.

Assume that $\mathbb{P}^0(A_\ell) > 0$ (or similary $\mathbb{P}^0(A_{-\ell}) > 0$)

Goal: $P^0(O_\ell) = 0$

Claim: $P^0(\exists n \geq 1, X_n \cdot \ell < 0) < 1$



never visits points with negative x-coord.

proof: if the claim is not true, then

$\mathbb{P}^0(\exists n \geq 1, X_n \cdot \ell < 0) = 1$

This implies that for any $x \in \mathbb{Z}^d$, $P^x(\exists n \geq 1, X_n \cdot \ell < 0) = 1$

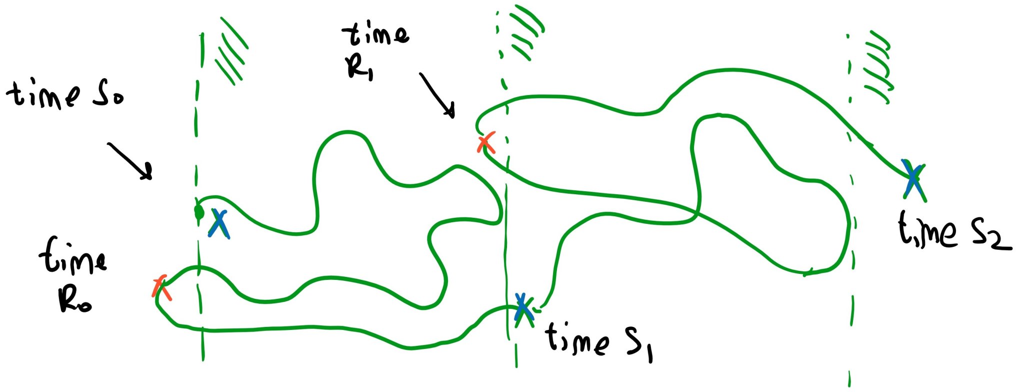
and this implies that $\mathbb{P}^0(\liminf_{n \to \infty} X_n \cdot \ell < 0) = 1$

and this contradicts the assumption that $\mathbb{P}^0(A_\ell)>0$.

To continue, denote $p = \mathbb{P}^0(\exists n\geq 1,\ X_n\cdot\ell<0) =$

for any $x\in\mathbb{Z}^d$ by trans-inv. $= \mathbb{P}^x(\exists n\geq 1,\ X_n\cdot\ell < x\cdot\ell)$

Idea: Define "regeneration times" - times at which the walk first enters a half space it never visited before



at each regeneration time, have probability p to go back left of that point and these are "bounded above by independent".

Let $S_0=0$ and inductively, for each k,

$$R_k := \min\{n>S_k : X_n\cdot\ell < X_{S_k}\cdot\ell\} \quad k\geq 0$$

$$S_k := \min\{n>R_{k-1} : X_n\cdot\ell > \max_{m\leq R_{k-1}} X_m\cdot\ell\} \quad k\geq 1$$

By definition, $\mathbb{P}^0(R_0<\infty)=p$.

$$\mathbb{P}(R_0<\infty, S_1<\infty, R_1<\infty) = \sum_{z\in\mathbb{Z}^d}\underbrace{\mathbb{P}^0(R_0<\infty, S_1<\infty, R_1<\infty, X_{S_1}=z)}_{(*)}$$

$$\circledast = \mathbb{E}_p[P^0_\omega(R_0<\infty, S_1<\infty, R_1<\infty, X_{S_1}=z)] \overset{\text{Strong Markov property}}{=}$$

$$= \mathbb{E}_p[\underbrace{P^0_\omega(R_0<\infty, S_1<\infty, X_{S_1}=z)}_{\substack{\text{a fcn. only of}\\ (\omega_x)_{x:\, x\cdot\ell < z\cdot\ell}}} \underbrace{P^z_\omega(\exists n\geq 1\ X_n\cdot\ell < z\cdot\ell)}_{\substack{\text{a fcn. only of}\\ (\omega_x)_{x:\, x\cdot\ell \geq z\cdot\ell}}}] \overset{P \text{ is IID}}{=}$$

$$= \underbrace{\mathbb{E}_p[P^0_\omega(R_0<\infty, S_1<\infty, X_{S_1}=z)]}_{\mathbb{P}^0(R_0<\infty, S_1<\infty, X_{S_1}=z)} \underbrace{\mathbb{E}_p[P^z_\omega(\exists n\geq 1\ X_n\cdot\ell < z\cdot\ell)]}_{\mathbb{P}^z(\exists n\geq 1\ X_n\cdot\ell < z\cdot\ell) = p}$$

In summary,

$$\mathbb{P}^0(R_0<\infty, S_1<\infty, R_1<\infty) = p\cdot \sum_{z\in\mathbb{Z}^d} \mathbb{P}^0(R_0<\infty, S_1<\infty, X_{S_1}=z) =$$

$$= p\cdot \mathbb{P}^0(R_0<\infty, S_1<\infty) \leq p\cdot \mathbb{P}^0(R_0<\infty) = p^2$$

By the same argument,

$$\mathbb{P}^0(R_0<\infty, S_1<\infty, \dots, R_k<\infty) \leq p^{k+1}$$

$\Longrightarrow$ $\mathbb{P}^0(\text{all } R_k \text{ and } S_k \text{ are finite}) = 0$

On O_ℓ, all R_k and S_k are finite so that $\mathbb{P}^0(O_\ell)=0$, as we wanted to prove

<u>We now show the Merkl-Zerner thm.:</u>

Planarity will be used to create intersections between random walk trajectories

Recall: Lévy's upward thm:

if $(\mathcal{F}_n)_{n\geq 0}$ is a filtration and A an event then,

$$\mathbb{P}(A|\mathcal{F}_n) \xrightarrow[n\to\infty]{} \mathbb{P}(A|\sigma(\bigcup_n \mathcal{F}_n)) \text{ a.s.}$$

We use this for the RW and the event A_ℓ. (we assume for simplicity that $\ell = (1,0)$)

$\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Then, for each ω and any $x \in \mathbb{Z}^d$

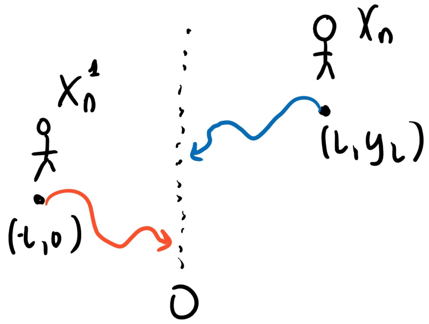
$$P_\omega^{X_n}(A_\ell) \underset{\text{Markov property}}{=} P_\omega^x(A_\ell|\mathcal{F}_n) \xrightarrow[n\to\infty]{(*)} P_\omega^x(A_\ell|\sigma(X_1, X_2 \dots)) = \mathbb{1}_{A_\ell}$$

Idea of the proof

Assume, to get a contradiction, that $P^0(A_\ell) > 0$ and $P^0(A_{-\ell}) > 0$. Recalling that then $P^0(A_\ell \cup A_{-\ell}) = 1$ by Kalikow's 0-1 law.

Start one random walker at $(-L, 0)$ and another one at (L, y_L)

After the first L steps, both walkers are nearly certain if A_ℓ happens for them or not



But since they're still in independent environments then these decisions are indep. and it is possible (since $P^0(A_\ell) > 0$, $P^0(A_{-\ell}) > 0$) that the one started at (L, y_L) will satisfy $A_{-\ell}$. and the one started at $(-L, 0)$ will satisfy A_ℓ.

On this event, it is very unlikely that they intersect

since: $P_\omega^{X_L^1}(A_\ell) \approx 1$ $P_\omega^{X_L^2}(A_\ell) \approx 0$ from $(*)$

Lastly, using planarity, it is shown that they intersect with a uniformly positive prob. when (y_L) is well chosen

We sketch the proof (not in every detail)

First, let us obtain a version of ⊛ with a uniform rate of convergence.

Claim 1: $\forall \varepsilon > 0 \ \exists M_\varepsilon > 0$ s.t. $\forall x \in \mathbb{Z}^d$,

$$P^x\left(|P_\omega^{X_n}(A_\ell) - \mathbb{1}_{A_\ell}| < \varepsilon \ \ \forall n > M_\varepsilon \right) \geq 1 - \varepsilon$$

Proof:

The prob. is the same for all x, so we prove for $x = 0$.

By ⊛ we know that $|P_\omega^{X_n}(A_\ell) - \mathbb{1}_{A_\ell}| < \varepsilon$ occurs for all $n > N_\varepsilon(\omega, (X_n))$ where $N_\varepsilon(\omega, (X_n)) < \infty$

Now, take M_ε so large s.t. $P^0(N_\varepsilon > M_\varepsilon) < \varepsilon$ □

We now consider a RW (X_n^1) starting at $(-L, 0)$ and another RW (X_n^2) starting at (L, y_L) for some y_L to be chosen later.

Call the annealed measure of both RWs by $\mathbb{P}_L$

first sample ω then run the 2 walks in ω

Claim 2: for any $(y_L)_{L \geq 1}$,

$$\mathbb{P}_L\left(\lim_{n\to\infty} X_n^1 \cdot \ell = +\infty, \lim_{n\to\infty} X_n^2 \cdot \ell = -\infty\right) \xrightarrow[L\to\infty]{} P^0(A_\ell) \cdot P^0(A_{-\ell})$$

Proof: If either $\mathbb{P}^0(A_\ell)=0$ or $\mathbb{P}^0(A_{-\ell})=0$ then, $\forall L$,
$\mathbb{P}_L(\lim_{n\to\infty} X_n^1\cdot\ell=\infty\ ,\ \lim_{n\to\infty} X_n^2\cdot\ell=-\infty)=0$ and there is nothing to prove. Assume then that both $\mathbb{P}^0(A_\ell)>0$ and $\mathbb{P}^0(A_{-\ell})>0$.

By Kalikow's 0-1 law we have $\mathbb{P}^0(A_\ell\cup A_{-\ell})=1$. It is simple to see that:

$$\left.\begin{aligned}&\lim_{L\to\infty}\mathbb{P}_L\big(\lim_{n\to\infty} X_n^1\cdot\ell=-\infty\ ,\ \exists n\ge1,\ X_n^1\cdot\ell\ge0\big)=0\\ &\lim_{L\to\infty}\mathbb{P}_L\big(\lim_{n\to\infty} X_n^2\cdot\ell=\infty\ ,\ \exists n\ge1,\ X_n^2\cdot\ell<0\big)=0\end{aligned}\right\}\ a$$

Hence, $\forall\varepsilon>0\ \exists L_\varepsilon$ s.t. $\forall L>L_\varepsilon$

$$\left|\mathbb{P}_L\big(\lim_{n\to\infty} X_n^1\cdot\ell=\infty,\ \lim_{n\to\infty} X_n^2\cdot\ell=-\infty\big)-\mathbb{P}_L\begin{pmatrix}\exists n\ge1\ X_n^1\cdot\ell\ge0\\ \exists m\ge1\ X_m^2\cdot\ell<0\end{pmatrix}\right|\le\varepsilon$$

Lastly,

$$\mathbb{P}_L\begin{pmatrix}\exists n\ge1\ X_n^1\cdot\ell\ge0\\ \exists m\ge1\ X_m^2\cdot\ell<0\end{pmatrix}=P\Big[\underbrace{P_\omega^{(-L,0)}(\exists n\ge1,X_n^1\cdot\ell\ge0)}_{\text{fcn. of }(\omega_x)_{x:\,x\cdot\ell<0}}\cdot\underbrace{P_\omega^{(L,y_2)}(\exists m\ge1,X_m^2\cdot\ell<0)}_{\text{fcn. of }(\omega_x)_{x:\,x\cdot\ell\ge0}}\Big]$$

$$\overset{P\ \text{IID}}{=}\mathbb{P}_L(\exists n\ge1, X_n^1\cdot\ell\ge0)\cdot\mathbb{P}_L(\exists m\ge1\ X_m^2\cdot\ell<0)\xrightarrow[L\to\infty]{}\mathbb{P}^0(A_\ell)\cdot\mathbb{P}^0(A_{-\ell})\ \text{ by } a$$

Claim 3: for any (y_L)

Possibly at different times (i.e., the traces intersect)

$$\lim_{L\to\infty}\mathbb{P}_L\Big(\lim_{n\to\infty} X_n^1\cdot\ell=+\infty,\ \lim X_n^2\cdot\ell=-\infty,\ \text{the walk trajectories intersect}\Big)=0$$

proof: It suffices to prove that:

$$\lim_{L\to\infty}\mathbb{P}_L\Big(\lim_{n\to\infty} X_n^1\cdot\ell=\infty,\ \lim_{n\to\infty} X_n^2\cdot\ell=-\infty,\ \begin{matrix}\text{the walk trajectories intersect}\\ \text{at some } x \text{ with } x\cdot\ell\le0\end{matrix}\Big)=0$$

The intersections necessarily happens after more than L steps of X_n^2.

Fix $\varepsilon>0$. Taking $L>M_\varepsilon$, we have that $|P_\omega^{X_n^2}(A_{-\ell})-\mathbb{1}_{A_{-\ell}}|<\varepsilon$ (**) for all $n>L$ with $\mathbb{P}_L$-Prob. at least $1-\varepsilon$. In particular, (**) occurs at the intersection point. Define a stopping time $T=\min\{n\ge0:\ P_\omega^{X_n}(A_\ell)<\varepsilon\}$ on the event of intersection and when (**) happens and $\lim_{n\to\infty} X_n^2\cdot\ell=-\infty$ then $T<\infty$.

Note that $P_\omega^{(-L,0)}(\lim_{n\to\infty} X_n^1\cdot\ell=\infty,\ T<\infty)<\varepsilon$. By strong Markov property at T.

In conclusion,

$$\mathbb{P}_L\Big(\lim_{n\to\infty} X^1_n\cdot\ell=\infty,\ \lim_{n\to\infty} X^2_n\cdot\ell=-\infty,\ \text{trajectories intersect at } x \text{ with } x\cdot\ell\le 0\Big)\le$$

$$\mathbb{P}_L\Big(\lim_{n\to\infty} X^1_n\cdot\ell=\infty,\ \lim_{n\to\infty} X^2_n\cdot\ell=-\infty,\ (\ast\ast)\ \text{holds, trajectories intersected at } x \text{ with } x\cdot\ell\le 0\Big)+\varepsilon\le$$

$$\le \mathbb{P}_L\Big(\lim_{n\to\infty} X^1_n\cdot\ell=\infty,\ T<\infty\Big)+\varepsilon\le 2\varepsilon.$$

which concludes the proof as ε is arbitrary.

It remains to "make the walks intersect" to get a contradiction

claim 4: For a specific (y_L)

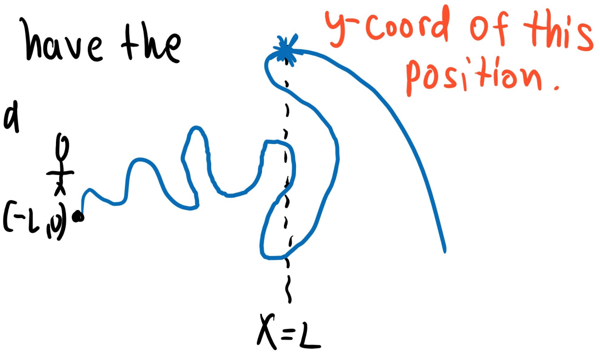
$$\mathbb{P}_L\Big(\lim_{n\to\infty} X_n\cdot\ell=+\infty,\ \lim_{n\to\infty} X_n\cdot\ell=-\infty,\ \text{no intersection of trajectories}\Big)\le$$

$$\le \frac{1}{2}\,\mathbb{P}^0(A_\ell)\,\mathbb{P}^0(A_{-\ell})$$

proof:

Choose y_L as the $\mathbb{P}_L$- median of the y-coord for the last visit of X^1_n to the line $X=L$, conditioned on $\lim_{n\to\infty} X^1_n\cdot\ell=+\infty$

By def, we have prob. $\ge\frac{1}{2}$ to have the last visit with y-coord $\ge y_L$ and prob. $\ge\frac{1}{2}$ for y-coord $\le y_L$



Homework exercise to prove that the claim holds with this choise of y_L.

Putting claims 2,3,4 together we conclude that $\mathbb{P}^0(A_\ell)\cdot\mathbb{P}^0(A_{-\ell})=0$ as required.